

Noncommutative instantons: a new approach.

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Abstract

We discuss instantons on noncommutative four-dimensional Euclidean space. In commutative case one can consider instantons directly on Euclidean space, then we should restrict ourselves to the gauge fields that are gauge equivalent to the trivial field at infinity. However, technically it is more convenient to work on four-dimensional sphere. We will show that the situation in noncommutative case is quite similar. One can analyze instantons taking as a starting point the algebra of smooth functions vanishing at infinity, but it is convenient to add a unit element to this algebra (this corresponds to a transition to a sphere at the level of topology). Our approach is more rigorous than previous considerations ; it seems that it is also simpler and more transparent. In particular, we obtain the ADHM equations in a very simple way.

Gauge theories on noncommutative spaces introduced by A.Connes [1] play now an important role in string/M-theory. (Their appearance was understood first from the viewpoint of Matrix theory in [2]; see [3], [4] for review of results obtained in this framework and [5] for the analysis in the framework of string theory.)

Instantons in noncommutative gauge theories were introduced in [6] for the case of noncommutative \mathbf{R}^4 and in [7] for the case of noncommutative T^4 . Later they were studied and applied in numerous papers (see [8], [9] for review). In present paper we suggest another approach to instantons on noncommutative \mathbf{R}^4 . It seems that it makes the theory much more transparent.

In commutative case one can study instantons directly on \mathbf{R}^4 ; then we should restrict ourselves to the gauge fields with finite euclidean action (or, equivalently to fields that are gauge equivalent to the trivial field at infinity). However, technically it is more convenient to work on four-dimensional sphere S^4 .

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We will show that the situation in noncommutative case is quite similar. The standard \mathbf{R}^4 is conformally equivalent to S^4 ; therefore the possibility to replace \mathbf{R}^4 with S^4 in the study of instantons is obvious. In noncommutative case we cannot apply this logic. However, we will see that it is useful to consider a unital algebra $\tilde{\mathbf{R}}_\theta^4$ obtained from the algebra \mathbf{R}_θ^4 by means of addition of unit element. Here \mathbf{R}_θ^4 stands for the algebra of smooth functions on noncommutative \mathbf{R}^4 that tend to zero at infinity (multiplication is defined as a star-product.) As usual in noncommutative geometry non-unital algebras are related to non-compact spaces; adjoining unit element corresponds to one-point compactification. This means that the transition from \mathbf{R}_θ^4 to $\tilde{\mathbf{R}}_\theta^4$ is analogous to transition from \mathbf{R}^4 to S^4 (at the level of topology).

Therefore one can think that it is easier to study instantons and solitons using $\tilde{\mathbf{R}}_\theta^4$; we'll see that this is true.

Physicists usually don't care of the precise definition of the space of functions they are working with (and usually this policy is completely justified). However, in the consideration of noncommutative \mathbf{R}^d it is important to be careful: different definitions of the space of functions on noncommutative space lead to different results. In the standard definition the algebra of functions on noncommutative \mathbf{R}^d is described as an algebra generated by noncommutative coordinates \hat{x}^k obeying $[\hat{x}^k, \hat{x}^l] = i\theta^{kl}$. We base our exposition of the theory of instantons on algebras \mathbf{R}_θ^d and $\tilde{\mathbf{R}}_\theta^d$ that don't contain \hat{x}^k at all. The absence of accurate definition of an instanton was not harmless, it led to a confusion in the question about existence of $U(1)$ -instantons. The present paper is not written as a rigorous mathematical paper, but it is based on accurate definitions (and it seems that the exposition can be made rigorous without essential difficulties).

The paper is organized as follows.

In Sec. 1 we consider instantons on noncommutative \mathbf{R}^4 using the algebra $\tilde{\mathbf{R}}_\theta^4$; we show that the noncommutative analog of ADHM data arises very naturally in this approach. The construction of noncommutative solitons also becomes simpler. In Sec.2 we show how to relate the consideration based on the algebra \mathbf{R}_θ^4 to the formalism of Sec. 1. In Sec. 3 we analyze the noncommutative analog of ADHM construction. In Sec. 4 we discuss some definitions and results of the preceding sections in more general setting and more accurately. We also compare definitions of instantons on noncommutative torus and on noncommutative Euclidean space. An appendix written by A. Connes contains a proof of the fact that modules used in present paper exhaust projective modules over appropriate algebras. (I could not find such a proof in the literature; I am indebted to A. Connes for giving a proof.)

Our paper does not depend on previous papers on noncommutative instantons, but we did not try to reproduce all known results using our approach. The necessary definitions of notions of noncommutative geometry [10] are given (but sometimes we omit some details). We use freely results of noncommutative geometry without explicit references; most of results we need are contained in [1], [11].

Preliminaries.

Let A be an associative algebra. A vector space E is a right A -module if we can multiply $e \in E$ by $a \in A$ from the right and this multiplication is distributive and associative (in particular, $(ea)b = e \cdot (ab)$ for $e \in E$, $a, b \in A$). Introducing the notation $\hat{a}e = ea$ we can say that to specify a right module we should assign to every $a \in A$ an operator $\hat{a} : E \rightarrow E$ in such a way that $\hat{a}\hat{b} = \hat{b}\hat{a}$. A linear map $\varphi : E \rightarrow E$ where E is an A -module is called an endomorphism if it is A -linear (i.e. $\varphi(ea) = \varphi(e)a$). The set of all endomorphisms can be considered as an algebra; it is denoted by $\text{End}_A E$. We say that A is an involutive algebra if it is equipped with antilinear involution $a \rightarrow a^+$ obeying $(ab)^+ = b^+a^+$. A module E over involutive algebra A is a (pre)Hilbert module if it is equipped with A -valued inner product $\langle e_1, e_2 \rangle$ obeying

$$\langle e_1 a_1, e_2 a_2 \rangle = a_1^+ \langle e_1, e_2 \rangle a_2.$$

for all $e_1, e_2 \in E$, $a_1, a_2 \in A$. If E_1, E_2 are two Hilbert A -modules we denote by $B(E_1, E_2)$ the set of all A -linear maps $T : E_1 \rightarrow E_2$ having adjoints. (One says that $T^* : E_2 \rightarrow E_1$ is a map adjoint to T if $\langle Tx, y \rangle = \langle x, T^*y \rangle$.) Notice that it follows from this definition that the spaces $B(E_1, E_2)$ and $B(E_2, E_1)$ are complex conjugate.

In a Hilbert module E we can construct endomorphisms by the following formula

$$\alpha(x) = \sum b_i \langle a_i, x \rangle$$

where $x, a_i, b_i \in E$. These endomorphisms are called endomorphisms of finite rank.

The set A^n consisting of column vectors with entries from A can be considered as a Hilbert A -module with respect to inner product

$$\langle a, b \rangle = \sum a_i^+ b_i.$$

In more invariant way we can define A^n as a tensor product of n -dimensional vector space V and the algebra A .

If A is a unital algebra the A -module A^n is called a free module with n generators. A projective module E over a unital algebra A is by definition a direct summand in A^n . (We consider only finitely generated projective modules.) Projective modules are Hilbert modules with respect to an inner product inherited from A^n . All endomorphisms of a projective module E have adjoints: $\text{End}_A E = B(E, E)$.

We gave all definitions for right modules, one can give similar definitions for left modules.

Section 1.

Let us consider the algebra \mathbf{R}_θ^d and the algebra $\tilde{\mathbf{R}}_\theta^d$ obtained from \mathbf{R}_θ^d by means of addition of unit element. Recall, that we consider elements of \mathbf{R}_θ^d as smooth functions on \mathbf{R}^d tending to zero at infinity. (More precisely, $f \in \mathbf{R}_\theta^d$

if derivatives of all orders exist and tend to zero at infinity). The product of functions $f, g \in \mathbf{R}_\theta^d$ (star-product) can be defined by the formula

$$(f \star g)(x) = \int \int f(x + \theta u) g(x + v) e^{iuv} du dv.$$

Operators $\alpha_v : f(x) \rightarrow f(x + v)$ where $v \in \mathbf{R}^d$ specify an action of Lie group \mathbf{R}^d on \mathbf{R}_θ^d . The derivatives $\partial_i = \partial/\partial x^i$ are infinitesimal automorphisms (derivations) of \mathbf{R}_θ^d . If E is a (right) A -module where $A = \mathbf{R}_\theta^d$ then a connection (a gauge field) on E is specified by means of \mathbf{C} -linear operators $\nabla_1, \dots, \nabla_d$ obeying the Leibniz rule:

$$\nabla_i(ea) = \nabla_i e \cdot a + e \cdot \partial_i a$$

where $e \in E$, $a \in A$.

The same definition can be used in the case $A = \tilde{\mathbf{R}}_\theta^d$ and in more general case when we have an action of Lie algebra \mathbf{R}^d on algebra A . If ∇', ∇ are two connections then the difference $\nabla'_i - \nabla_i$ commutes with multiplication by elements $a \in A$; in other words $\nabla'_i - \nabla_i$ is an A -linear map or an endomorphism of the module E . It is convenient to fix a connection $\nabla^{(0)}$ on E ; then every other connection can be represented in the form

$$\nabla_i = \nabla_i^{(0)} + \alpha_i,$$

where $\alpha_i \in \text{End}_A E$ are elements of the algebra $\text{End}_A E$ of endomorphisms of the module E .

Every algebra A can be considered as a module over itself. More precisely the multiplication from the right gives A a structure of a right A -module that will be denoted by A^1 ; to obtain a left module we should consider the multiplication from the left. It follows from associativity that the multiplication from the left commutes with multiplication from the right: $(ae)b = a(eb)$. This means that the operators of left multiplication $\varphi_a : e \rightarrow ae$ are endomorphisms of A^1 (of A considered as a right module). If A is a unital algebra every endomorphism $\varphi : A^1 \rightarrow A^1$ has this form: $\varphi(e) = \varphi(1 \cdot e) = \varphi(1) \cdot e$. This statement can be applied to unital algebra $\tilde{\mathbf{R}}_\theta^d$, but for the algebra \mathbf{R}_θ^d we have also other endomorphisms of $(\mathbf{R}_\theta^d)^1$. Namely, endomorphisms of $(\mathbf{R}_\theta^d)^1$ correspond to smooth bounded functions on \mathbf{R}^d : every function $\alpha(x)$ of this kind determines an endomorphism transforming $e(x) \in (\mathbf{R}_\theta^d)^1$ into star-product $\alpha(x) \star e(x)$. (Talking about smooth bounded functions we have in mind functions having bounded derivatives $\partial_\alpha f$ where $\alpha = (\alpha_1, \dots, \alpha_d)$ is a multiindex.)

Derivatives $\partial_1, \dots, \partial_d$ satisfy the Leibniz rule and therefore specify a connection on A^1 for $A = \mathbf{R}_\theta^d$ or $A = \tilde{\mathbf{R}}_\theta^d$. All other connections have the form

$$\nabla_i e(x) = \partial_i e(x) + \alpha_i(x) \star e(x)$$

where $\alpha_1(x), \dots, \alpha_d(x)$ are smooth functions that are bounded in the case of \mathbf{R}_θ^d and tend to a constant at infinity in the case of $\tilde{\mathbf{R}}_\theta^d$.

A little bit more complicated module A^n can be obtained if we take a direct sum of n copies of the module A^1 . Elements of A^n can be considered as column vectors with entries from A . Endomorphisms of A^n can be identified with $n \times n$ matrices having endomorphisms of A^1 as their entries. Every connection on $A^n = (\mathbf{R}_\theta^d)^n$ or $A^n = (\tilde{\mathbf{R}}_\theta^d)^n$ can be represented in the form

$$\nabla_i e_k(x) = \partial_i e_k(x) + (\alpha_i(x))_k^l \star e_l(x) \quad (1)$$

where the entries of $n \times n$ matrices $\alpha_1(x), \dots, \alpha_d(x)$ obey the same conditions as functions $\alpha_i(x)$ in the expression for the connection in A^1 . Algebras \mathbf{R}_θ^d and $\tilde{\mathbf{R}}_\theta^d$ are involutive algebras with respect to complex conjugation. This means that we can consider Hilbert modules over these algebras. We'll identify gauge fields with unitary connections, i.e. connections represented by operators ∇_i obeying the condition

$$\langle \nabla_i a, b \rangle + \langle a, \nabla_i b \rangle = \partial_i \langle a, b \rangle.$$

Field strength of a gauge field $F_{ij} = [\nabla_i, \nabla_j]$ obeys similar condition.

Modules we considered were Hilbert modules. We can construct also other Hilbert modules in the following way.

Let us consider Hermitian operators \hat{x}^k satisfying commutation relations

$$[\hat{x}^k, \hat{x}^l] = i\theta^{kl} \cdot 1 \quad (2)$$

We will assume, that $d = 2n$ and θ is a nondegenerate $2n \times 2n$ matrix; then these relations are equivalent to canonical commutation relations $[\hat{x}^k, \hat{x}^{k+n}] = i$, $[\hat{x}^k, \hat{x}^l] = 0$ if both $k, l \leq n$ or $k, l > n$. (More precisely, we can write (2) in the canonical form replacing \hat{x}^i by their linear combinations.) It is well known (Stone-von Neumann theorem) that irreducible representation of canonical commutation relations is unique and that one can obtain all other representations taking direct sums of several copies of irreducible representations. By means of Hermitian operators \hat{x}^k specifying an irreducible representation of (2) we can assign to every function $\varphi(x) = \int \varphi(k) e^{ikx} dk$ an operator $\hat{\varphi} = \int \varphi(k) e^{ik\hat{x}} dk$. Taking $\varphi \in \mathbf{R}_\theta^d$ we obtain an \mathbf{R}_θ^d -module \mathcal{F} . Notice that for $f \in S(\mathbf{R}^n)$, $\varphi \in \mathbf{R}_\theta^d$ we have $\hat{\varphi}f \in S(\mathbf{R}^n)$ (see Sec. 4). This means that $S(\mathbf{R}^n)$ can be regarded as an irreducible \mathbf{R}_θ^d -module; we will denote it by the same letter \mathcal{F} . Every \mathbf{R}_θ^d -module can be considered as $\tilde{\mathbf{R}}_\theta^d$ -module (the unit element is represented by identity operator), therefore we can regard \mathcal{F} as an $\tilde{\mathbf{R}}_\theta^d$ -module.

The module \mathcal{F} is a projective $\tilde{\mathbf{R}}_\theta^d$ -module. This is clear from the following construction of it that is more convenient for us. Let us fix a real function $p \in \mathbf{R}_\theta^d$ obeying $p \star p = p$ and $\int p dx = 1$. (In the correspondence between elements of \mathbf{R}_θ^d and operators such a function corresponds to one-dimensional projection \hat{p} .) Then \mathcal{F} can be defined as a submodule of $\tilde{\mathbf{R}}_\theta^d$ consisting of elements of the form $p \star r$ where $r \in \tilde{\mathbf{R}}_\theta^d$. (If \hat{p} is a projection on one-dimensional space spanned by a normalized vector α matrix elements of \hat{p} are given by the formula $\langle x | \hat{p} | x' \rangle = \bar{\alpha}(x)\alpha(x')$ and matrix elements of the operator corresponding to

$p \star r$ have the form $\bar{\alpha}(x)\rho(x')$ for some function $\rho \in S(\mathbf{R}^n)$.) The free module $\tilde{\mathbf{R}}_\theta^d$ is a direct sum of a module \mathcal{F} and a module consisting of elements of the form $(1-p) \star r$ therefore \mathcal{F} is projective. It follows from this remark that the module \mathcal{F} can be characterized also as a submodule consisting of elements v obeying $(1-p) \star v = 0$. Replacing the condition $\int p dx = 1$ by the condition $\int p dx = k \in \mathbf{Z}$ we obtain by means of the same construction a module isomorphic to \mathcal{F}^k . (The corresponding operator \hat{p} is a projection on a k -dimensional space E_k and has matrix elements of the form $\langle x | \hat{p} | x' \rangle = \bar{\alpha}_1(x)\alpha_1(x') + \dots + \bar{\alpha}_k(x)\alpha_k(x')$ where $\alpha_1, \dots, \alpha_k$ stands for orthonormal basis of E_k . Matrix elements of the operator corresponding to $p \star r$ have the form $\bar{\alpha}_1(x)\rho_1(x') + \dots + \bar{\alpha}_k(x)\rho_k(x')$ for some functions ρ_1, \dots, ρ_k .)

Similar statement is true for projections on $(\mathbf{R}_\theta^d)^s$. Considering $\int p dx$ as a trace on \mathbf{R}_θ^d we define a trace on matrices with entries from \mathbf{R}_θ^d . If Π is a self-adjoint $s \times s$ matrix of this kind, $\Pi^2 = \Pi$ and $\text{Tr} \Pi = k$ then the matrices of the form ΠR , where R is an $s \times s$ matrix with entries from $\tilde{\mathbf{R}}_\theta^d$ constitute an $\tilde{\mathbf{R}}_\theta^d$ -module isomorphic to \mathcal{F}^k . Here multiplication of matrices is understood as a combination of the usual matrix product and the star-product. A matrix $V = \Pi R$ obeys $(1 - \Pi)V = 0$. This remark leads to an alternative definition of the module at hand: considering $1 - \Pi$ as an operator acting on $(\mathbf{R}_\theta^d)^s$ we can say that $\text{Ker}(1 - \Pi)$ is isomorphic to \mathcal{F}^k .

Operators $\nabla_k^{(0)}$ related with \hat{x}^l by the formula $\hat{x}^k = i\theta^{kl}\nabla_l^{(0)}$ specify a connection on the module \mathcal{F} . It follows from Schur's lemma that all endomorphisms of \mathcal{F} have the form $\text{const} \cdot 1$; therefore all other connections can be represented as $\nabla_k = \nabla_k^{(0)} + c_k \cdot 1$. We'll study $\tilde{\mathbf{R}}_\theta^d$ -modules \mathcal{F}_{rs} that are represented as direct sums of r copies of the module \mathcal{F} and s copies of the module $(\tilde{\mathbf{R}}_\theta^d)^1$. All these modules are projective. One can prove that every projective $\tilde{\mathbf{R}}_\theta^d$ -module is isomorphic to one of modules \mathcal{F}_{rs} (see Sec.4 and Appendix).

Every connection on \mathcal{F}_{rs} can be written in the form

$$\nabla_k = \nabla_k^{(0)} + \Phi, \quad \Phi = \begin{pmatrix} M_k & N_k \\ S_k & T_k \end{pmatrix} \quad (3)$$

where $\nabla_k^{(0)}$ is the standard connection acting as $i(\theta^{-1})_{kl}\hat{x}^l$ on \mathcal{F} and as ∂_k on $(\mathbf{R}_\theta^d)^1$, M_k is an $r \times r$ matrix with entries from \mathbf{C} , N_k is an $r \times s$ matrix with entries from \mathcal{F} , S_k is an $s \times r$ matrix with entries from $\tilde{\mathcal{F}}$ and T_k is an $s \times s$ matrix with entries from $\tilde{\mathbf{R}}_\theta^d$. (Here $\tilde{\mathcal{F}}$ is obtained from \mathcal{F} by means of complex conjugation.) This follows from the remark that an endomorphism of direct sum $E_1 + \dots + E_r$ where E_i are A -modules can be described as a matrix having an A -linear map from E_k in E_l as an element in k -th row and l -th column. (For any unital algebra A an A -linear map from A^1 into any A -module E is characterized by the image of unit element; this means that the space of A -linear maps $A^1 \rightarrow E$ can be identified with E . The space of $\tilde{\mathbf{R}}_\theta^d$ -linear maps from \mathcal{F} into $\tilde{\mathbf{R}}_\theta^d$ can be identified with $B(\mathcal{F}, \tilde{\mathbf{R}}_\theta^d)$; it is complex conjugate to $B(\tilde{\mathbf{R}}_\theta^d, \mathcal{F}) = \mathcal{F}$.)

It is easy to see that the connection $\nabla_k^{(0)}$ satisfies the Yang-Mills equations of motion. Let us consider Yang-Mills field interacting with scalar field φ in adjoint representation; if Yang-Mills field is regarded as a connection on a module \mathcal{F}_{rs} then φ should be regarded as an endomorphism of this module. We denote by φ_0 an endomorphism of \mathcal{F}_{rs} acting as an identity map on \mathcal{F}^r and as a zero map on $\tilde{\mathbf{R}}_\theta^s$. Taking the gauge field as $\nabla_k^{(0)}$ and the scalar field as $\alpha\varphi_0$ we obtain a solution to the equations of motion if α and zero are stationary point of the potential of the scalar field. It is easy to check that this simple solution corresponds to the solution of [13]; see Sec 2. This is an additional confirmation of the idea that it is necessary to use not only free modules, but also other modules; a consideration of classical solutions to equations of motion based on this idea will be given in [14].

Now we can turn to the study of instantons on $\tilde{\mathbf{R}}_\theta^4$ -modules \mathcal{F}_{rs} . By definition, an instanton is a gauge field (unitary connection) obeying $F^+ = 0$ where F^+ is the self-dual part of the curvature $F_{kl} = [\nabla_k, \nabla_l]$. It is convenient to introduce operators

$$\begin{aligned} D_1 &= \nabla_1 + i\nabla_2, & D_2 &= \nabla_3 + i\nabla_4, \\ D_1^+ &= -\nabla_1 + i\nabla_2, & D_2^+ &= \nabla_3 + i\nabla_4 \end{aligned} \quad (4)$$

Then an instanton satisfies

$$[D_1, D_2] = 0, \quad (5)$$

$$[D_1, D_1^+] + [D_2, D_2^+] = 0. \quad (6)$$

Let us represent D_1 and D_2 in the form

$$D_1 = D_1^{(0)} + \begin{pmatrix} B_1 & I \\ K & R_1 \end{pmatrix}, \quad D_2 = D_2^{(0)} + \begin{pmatrix} B_2 & L \\ J & R_2 \end{pmatrix}. \quad (7)$$

We'll assume that $K = 0$ and $L = 0$ (these conditions can be considered as gauge conditions). Then the equation (5) can be rewritten in the following way

$$\alpha \cdot 1 + [B_1, B_2] + IJ = 0, \quad (8)$$

$$IR_2 - B_2I - D_2^{(0)}I = 0, \quad (9)$$

$$R_1J - JB_1 + D_1^{(0)}J = 0, \quad (10)$$

$$-JI + [R_1, R_2] + \partial_1^{(0)}R_2 - \partial_2^{(0)}R_1 = 0. \quad (11)$$

Multiplication in these formulas is considered as the usual matrix multiplication combined with natural bilinear maps $\mathcal{F} \otimes \tilde{\mathcal{F}} \rightarrow \mathbf{C}$, $\mathcal{F} \otimes \tilde{\mathbf{R}}_\theta^d \rightarrow \mathcal{F}$, $\tilde{\mathbf{R}}_\theta^d \otimes \tilde{\mathcal{F}} \rightarrow \tilde{\mathcal{F}}$, $\tilde{\mathbf{R}}_\theta^d \otimes \tilde{\mathbf{R}}_\theta^d \rightarrow \tilde{\mathbf{R}}_\theta^d$, $\tilde{\mathcal{F}} \otimes \mathcal{F} \rightarrow \tilde{\mathbf{R}}_\theta^d$. We use these maps for $d = 4$, but they are defined for every even d and nondegenerate θ . The first of these maps stems from hermitian inner product, the second from action of $\tilde{\mathbf{R}}_\theta^d$ on \mathcal{F} . The last map is an $\tilde{\mathbf{R}}_\theta^d$ -valued inner product on \mathcal{F} (the module \mathcal{F} is projective and therefore can be

considered as a Hilbert module). The operators $D_i^{(0)}, \partial_i^{(0)}$, $i = 1, 2$ are related to $\nabla_i^{(0)}, \partial_i$, $i = 1, 2, 3, 4$ in the same way as D_1, D_2 are related to ∇_i . One can express $\alpha \cdot 1 = [D_1^{(0)}, D_2^{(0)}]$ in terms of θ^{jk} (namely, $\alpha = \theta^{13} - \theta^{24} + i\theta^{23} + i\theta^{14}$). The equation (6) also can be represented by equations for four blocks; we write down only the first one:

$$\beta \cdot 1 + [B_1, B_1^+] + [B_2, B_2^+] + II^+ - J^+ J = 0 \quad (12)$$

where $\beta \cdot 1 = [D_1^{(0)}, D_1^{(0)+}] + [D_2^{(0)}, D_2^{(0)+}]$. It is easy to see that equations (8) and (12) are closely related to ADHM construction; moreover, they coincide with noncommutative counterpart of ADHM equations if we make an ansatz $I = I^{(0)} \cdot \Phi$, $J = J^{(0)} \cdot \Psi$ where $I^{(0)}$ and $J^{(0)}$ are matrices with complex entries, $\Phi \in \mathcal{F}$, $\Psi \in \bar{\mathcal{F}}$, $\Phi \Psi = 1$. It is natural to conjecture that having a solution of noncommutative ADHM equations (8), (12) we can find Φ, Ψ, R_1, R_2 in such a way that (5), (6) are satisfied.

The above consideration can be applied with minor modifications to instantons on noncommutative orbifold $\mathbf{R}^4_\theta/\Gamma$; it leads to noncommutative analog of equivariant ADHM equations found in [12].

Section 2.

In this section we'll consider instantons on noncommutative \mathbf{R}^4 in terms of the algebra \mathbf{R}^4_θ . We'll relate this consideration to the analysis in terms of $\tilde{\mathbf{R}}^4_\theta$ given in Sec. 1.

Let us recall the standard definition of instanton on commutative \mathbf{R}^4 . In this definition we consider solutions of (anti)selfduality equation, that are gauge trivial at infinity. More precisely we restrict ourselves to gauge fields that can be represented at infinity in the form

$$A_\mu \approx g^{-1}(x) \partial_\mu g(x) \quad (13)$$

where $g(x)$ is a function taking values in the gauge group G . The function $g(x)$ is defined outside some ball D and cannot be extended to the whole space \mathbf{R}^4 (the obstruction to this extension can be identified with topological number of the gauge field). To find an appropriate condition of "triviality at infinity" on noncommutative \mathbf{R}^4 we start with reformulation of this condition in the commutative case. We assume that $G = U(n)$ and consider matrix valued functions $g(x)$, $h(x)$ satisfying the condition

$$h(x)g(x) = 1 - p(x)$$

where $p(x)$ rapidly tends to zero at infinity. Then a gauge field A_μ is trivial at infinity if it can be represented in the form

$$A_\mu(x) = h(x) \partial_\mu g(x) + \alpha_\mu(x) \quad (14)$$

where α_μ tends to zero at infinity faster than $\|x\|^{-1}$. (If A_μ is represented in the form (13) we can obtain a representation in the form (14) taking as

$g(x)$, $h(x)$ extensions of $g(x)$, $g^{-1}(x)$ to the whole space \mathbf{R}^4 . Such extensions don't exist if we consider $g(x)$, $g^{-1}(x)$ as $U(n)$ -valued functions, but they do exist if we consider them as matrix valued functions.)

It is convenient to represent (14) in the form

$$\nabla_\mu = \hat{h} \circ \partial_\mu \circ \hat{g} + \hat{\beta}_\mu \quad (15)$$

where $\hat{h}, \hat{g}, \hat{\beta}_\mu$ stand for operators of multiplication by h, g, β_μ . We'll use this representation to generalize the above considerations for the gauge fields on noncommutative \mathbf{R}^4 . We consider these gauge fields as connections on the module $(\mathbf{R}_\theta^4)^n$. We have seen that such connections can be represented in the form (1), where α_i belong to the algebra $B = \text{End}_{\mathbf{R}_\theta^4}(\mathbf{R}_\theta^4)^n$ of endomorphisms of $(\mathbf{R}_\theta^4)^n$ (they can be considered as $n \times n$ matrices having bounded smooth functions as their entries).

Let us consider the subalgebra $B_0 \subset B$ consisting of all endomorphisms tending to zero at infinity faster than $\|x\|^{-1}$. More precisely, elements of B_0 should be represented by matrices with entries belonging to the algebra Γ_ρ^m with $m < -1$. (See Sec.4 for the definition of Γ_ρ^m .) This means that B_0 can be regarded as a subalgebra of the algebra of endomorphisms of the $\tilde{\mathcal{A}}$ -module $(\tilde{\mathcal{A}})^n$ where we can take $\mathcal{A} = \mathbf{R}_\theta^4$ or $\mathcal{A} = \Gamma_\rho^m$, $m < -1$.

Let us say that a gauge field ∇_μ on \mathbf{R}_θ^4 is gauge trivial at infinity if

$$\nabla_\mu = T \circ \partial_\mu \circ T^+ + (1 - TT^+) \circ \partial_\mu + \sigma_\mu \quad (16)$$

where $T^+T = 1$, $\sigma_\mu \in B_0$ and T^+ is a parametrix of T , i.e. $1 - TT^+ = \Pi$ is a matrix with entries from the Schwartz space $S(\mathbf{R}^4)$. Here $T \in B = \text{End}_{\mathbf{R}_\theta^4}(\mathbf{R}_\theta^4)^n$ is an $n \times n$ matrix considered as an endomorphism of $(\mathbf{R}_\theta^4)^n$. (See Sec. 4 for the conditions of existence of parametrix.)

Having an endomorphism T obeying the above conditions we can show that $\text{Ker}T^+ \oplus (\mathbf{R}_\theta^4)^n$ is isomorphic to $(\mathbf{R}_\theta^4)^n$ as a Hilbert \mathbf{R}_θ^4 -module. The construction of the isomorphism goes as follows. We map $(\xi, x) \in \text{Ker}T^+ \oplus (\mathbf{R}_\theta^4)^n$ into $\xi + Tx \in (\mathbf{R}_\theta^4)^n$. The inverse map transforms $y \in (\mathbf{R}_\theta^4)^n$ into $(\Pi y, T^+y)$. (The map $\Pi = 1 - TT^+$ is a projector of $(\mathbf{R}_\theta^4)^n$ onto $\text{Ker}T^+$). These maps preserve \mathbf{R}_θ^4 -valued inner products: $\langle \xi + Tx, \xi' + Tx' \rangle = \langle \xi, \xi' \rangle + \langle Tx, \xi' \rangle + \langle \xi, Tx' \rangle + \langle Tx, Tx' \rangle = \langle \xi, \xi' \rangle + \langle x, T\xi' \rangle + \langle T^+\xi, x' \rangle + \langle T^+Tx, x' \rangle = \langle \xi, \xi' \rangle + \langle x, x' \rangle$. It follows from $\Pi \in S(\mathbf{R}^4)$ that the \mathbf{R}_θ^4 -module $\text{Ker}T^+$ is isomorphic to \mathcal{F}^k for some k . (We are using the fact that $\text{Ker}TT^+ = \text{Ker}T^+$ and that $\text{Ker}(1 - \Pi) = \text{Ker}TT^+$ is isomorphic to \mathcal{F}^k .) We can conclude therefore that $\mathcal{F}^k \otimes (\mathbf{R}_\theta^4)^n$ is isomorphic to $(\mathbf{R}_\theta^4)^n$ as Hilbert \mathbf{R}_θ^4 -module.

Let us analyze the relation between connections on modules $(\mathbf{R}_\theta^4)^n$ and $\text{Ker}T^+ \oplus (\mathbf{R}_\theta^4)^n$.

The trivial connection ∂_α on $(\mathbf{R}_\theta^4)^n$ induces a connection $\Pi\partial_\alpha\Pi$ on $\text{Ker}T^+$. Let us calculate the connection D_α on $(\mathbf{R}_\theta^4)^n$ that corresponds to the connection

$(\Pi\partial_\alpha\Pi, \partial_\alpha)$ on $\text{Ker}T^+ \oplus (\mathbf{R}_\theta^4)^n$. We obtain $y \rightarrow (\Pi y, T^+y) \rightarrow (\Pi\partial_\alpha\Pi y, \partial_\alpha T^+y) \rightarrow \Pi\partial_\alpha\Pi y + T\partial_\alpha T^+$ hence

$$D_\alpha = T\partial_\alpha T^+ + \Pi\partial_\alpha\Pi = T_\alpha\partial_\alpha T^+ + \Pi\partial_\alpha + \rho_\alpha \quad (17)$$

where $\rho_\alpha = \Pi[\partial_\alpha, \Pi]$. We see that D_α is gauge trivial at infinity. Gauge fields on $(\mathbf{R}_\theta^4)^n$ that are gauge trivial at infinity (i.e. have the form (16)) correspond on $\text{Ker}T^+ \oplus (\mathbf{R}_\theta^4)^n$ to gauge fields of the form

$$(\Pi\partial_\alpha\Pi, \partial_\alpha) + \nu_\alpha \quad (18)$$

where ν_α have the same behavior at infinity as σ_α in (16). More precisely, if $\sigma_\alpha \in \Gamma_\rho^m$ then the field (18) can be extended to $\tilde{\Gamma}_\rho^m$ -module \mathcal{F}_{kn} (and, because $\Gamma_\rho^m \subset \mathbf{R}_\theta^4$, also to \mathcal{F}_{kn} considered as $\tilde{\mathbf{R}}_\theta^4$ -module).

It is clear that in the above consideration we can replace \mathbf{R}_θ^4 with \mathbf{R}_θ^d . (The matrix θ should be nondegenerate, hence d is necessarily even.) Endomorphisms obeying $T^+T = 1$ (isometries) were used in several papers to construct noncommutative solitons (see, for example, [8], [9], [13], [16], [17]). The approach of these papers is closely related to our treatment. In particular, the simple solution to the equation of motion for Yang-Mills field interacting with scalar field described in Sec. 1 can be transformed into noncommutative soliton of [13] by means of the isomorphism between $\text{Ker}T^+ \otimes (\mathbf{R}_\theta^{2k})^n$ and $(\mathbf{R}_\theta^{2k})^n$. The standard connection $\nabla_\alpha^{(0)}$ of Sec.1 is an instanton if the matrix $\theta_{\alpha\beta}$ is antiselfdual; it corresponds to the instanton found in [17]. Now we are able to compare different definitions of instantons. One can consider instantons as (anti)selfdual gauge fields on $(\mathbf{R}_\theta^4)^n$ that are gauge trivial at infinity. It follows from the above consideration that equivalently we can consider instantons as gauge fields (18) on $\mathcal{F}^k \oplus (\mathbf{R}_\theta^4)^n$ obeying (anti)selfduality equation. Using the fact that fields (18) can be extended to $\tilde{\mathbf{R}}_\theta^4$ -module \mathcal{F}_{kn} we obtain the relation between instantons on $(\mathbf{R}_\theta^4)^n$ and instantons studied in Sec. 1. Namely, we see that instantons on \mathcal{F}_{rs} (the fields (7) obeying equations (5),(6)) correspond to instantons on $(\mathbf{R}_\theta^4)^s$ if R_1, R_2 are matrices with entries from Γ_ρ^m , $m < -1$.

Section 3.

Let us analyze the noncommutative analog of ADHM construction [1], [3] in our approach. We consider an \mathbf{R}_θ^4 -linear operator

$$D^+ : (V \oplus V \oplus W) \otimes \mathbf{R}_\theta^4 \rightarrow (V \oplus V) \otimes \mathbf{R}_\theta^4$$

defined by the formula

$$D^+ = \begin{pmatrix} -B_2 + z_2 & B_1 - z_1 & I \\ B_1^+ - \bar{z}_1 & B_2^+ - \bar{z}_2 & J^+ \end{pmatrix}. \quad (19)$$

Here $B_1, B_2 \in \text{Hom}(V, V)$, $I \in \text{Hom}(W, V)$, $J \in \text{Hom}(V, W)$, $z_1 = \hat{x}^1 + i\hat{x}^2$, $z_2 = \hat{x}^3 + i\hat{x}^4$, $[x^r, x^s] = i\theta^{rs}$. Commutation relations between z_i, \bar{z}_i can

be written in the form

$$[z_1, z_2] = -\zeta_c, [z_1, \bar{z}_1] = -\zeta_1, [z_2, \bar{z}_2] = -\zeta_2$$

(We use the notations of [3].) We impose the condition that the operator

$$D^+ D : (V \oplus V) \otimes \mathbf{R}_\theta^4 \rightarrow (V \oplus V) \otimes \mathbf{R}_\theta^4$$

has the form

$$\begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix} \quad (20)$$

where $\Delta : V \otimes \mathbf{R}_\theta^4 \rightarrow V \otimes \mathbf{R}_\theta^4$ can be written as

$$\begin{aligned} \Delta &= (B_1 - z_1)(B_2^+ - \bar{z}_1) + (B_2 - z_2)(B_2^+ - \bar{z}_2) + II^+ \\ &= (B_1^+ - \bar{z}_1)(B_1 - z_1) + (B_2^+ - \bar{z}_2)(B_2 - z_2) + J^+ J. \end{aligned}$$

It is easy to check that this condition is equivalent to the noncommutative ADHM equations (8), (12). One can check that the operator Δ has no zero modes (see [3]).

Let us analyze the space $\mathcal{E} \subset (V \oplus V \oplus W) \otimes \mathbf{R}_\theta^4$ consisting of solutions to the equation $D^+ \psi = 0$. From \mathbf{R}_θ^4 -linearity of D^+ it follows that \mathcal{E} can be considered as \mathbf{R}_θ^4 -module.

One can consider an orthogonal projection P of the module $(V \oplus V \oplus W) \otimes \mathbf{R}_\theta^4$ onto \mathcal{E} ; the standard connection ∂_α on $(V \oplus V \oplus W) \otimes \mathbf{R}_\theta^4$ generates an instanton $P \circ \partial_\alpha \circ P$ on \mathcal{E} (see [1], [3]). An equivalent way to construct an instanton is to consider an isometric \mathbf{R}_θ^4 -linear map $\omega : \mathcal{E}' \rightarrow (V \oplus V \oplus W) \otimes \mathbf{R}_\theta^4$ having \mathcal{E} as its image and to define connection on \mathcal{E}' by the formula $\omega^+ \circ \partial_\alpha \circ \omega$. (Here \mathcal{E}' is a Hilbert module, a map is isometric if $\omega^+ \omega = 1$.)

We will relate this construction to the above consideration finding explicitly the module \mathcal{E}' and the map ω . This will allow us to say that gauge fields obtained by means of noncommutative ADHM construction are instantons in the sense of present paper.

Representing ψ as a column vector

$$\begin{pmatrix} \varphi \\ \zeta \end{pmatrix} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \zeta \end{pmatrix} \quad (21)$$

where $\varphi_i \in V \otimes \mathbf{R}_\theta^4$, $\xi \in W \otimes \mathbf{R}_\theta^4$ we can rewrite the equation $D_+ \psi = 0$ in the form

$$A\varphi + C\xi = 0 \quad (22)$$

where

$$A\varphi = \begin{pmatrix} -B_2 + z_2 & B_1 - z_1 \\ B_1^+ - \bar{z}_1 & B_2^+ - \bar{z}_2 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \quad C\xi = \begin{pmatrix} I\xi \\ J^+\xi \end{pmatrix} \quad (23)$$

We will assume that θ is a nondegenerate matrix; then without loss of generality we can assume that $\zeta_1 > 0$, $\zeta_2 > 0$. Our assumption allows us to consider elements of \mathbf{R}_θ^4 as pseudodifferential operators acting on functions defined on \mathbf{R}^2 . Then all above equations can be regarded as operator equations. In particular, A is a first order differential elliptic operator of index $r = \dim V$ and has a parametrix Q , a pseudodifferential operator obeying $QA = 1 - \Pi$, $AQ = 1 - \Pi'$ where Π, Π' are operators of finite rank.

The operator equation (22) can be reduced to the equation

$$Af + Cg = 0 \quad (24)$$

where f and g are functions defined on \mathbf{R}^2 . If \mathcal{N} is the space of solutions to (24) then solutions to (22) can be characterised as pseudodifferential operators taking values in \mathcal{N} . Using the parametrix we can reduce the study of Eqn (24) to the analysis of equation $(1 - \Pi)f + QCg = 0$; this is essentially a finite-dimensional problem. We will assume that $A \cdot Q = 1$ (i. e. $\Pi' = 0$); this technical assumption leads to some simplification of calculations. In our conditions the operator A has a kernel $\text{Ker} A$ of dimension $r = \dim V$. Every solution of (24) can be written as $F - QCg$ where $F \in \text{Ker} A$. This permits us to say that the \mathbf{R}_θ^4 -module \mathcal{E} is isomorphic to the direct sum

$$\mathcal{E}' = \mathcal{F}^r \otimes (W \otimes \mathbf{R}_\theta^4).$$

An \mathbf{R}_θ^4 -linear map $v : \mathcal{E}' \rightarrow (V \oplus V \oplus W) \otimes \mathbf{R}_\theta^4$ defined by the formula

$$v \begin{pmatrix} \rho \\ \xi \end{pmatrix} = \begin{pmatrix} \rho - QC\xi \\ \xi \end{pmatrix} \quad (25)$$

is an isomorphism between \mathcal{E}' and \mathcal{E} . (We consider \mathcal{F}^r as a space consisting of pseudodifferential operators $\rho \in (V \oplus V) \otimes \mathbf{R}_\theta^4$ taking values in $\text{Ker} A$; if f_1, \dots, f_r is a basis of $\text{Ker} A$ then $\langle x | \rho | x' \rangle = \sum_{1 \leq k \leq r} f_k(x) \rho_k(x')$.) An isometric map $w : \mathcal{E}' \rightarrow \mathcal{E}$ can be constructed as $w = v(v^+v)^{-1/2}$.

To prove that the map v is an isomorphism and to check that the gauge field $\omega^+ \circ \partial_j \circ \omega$ is an instanton we need some information that can be obtained easily from well known results about pseudodifferential operators [15]; necessary results are formulated in Sec.4. Notice that at infinity ω has the same behavior as v . This follows from the remark that $v^+v - 1$ tends to zero at infinity as $\|z\|^{-1}$. (Using the notations of Sec.4 we have $A \in H\Gamma_1^{1,1}$, $C \in \Gamma_1^0$, $Q \in H\Gamma_1^{-1,-1}$, hence $v^+v - 1 \in \Gamma_1^{-1}$.) This remark permits us to say that the asymptotic behavior of $\omega^+ \circ \partial \circ \omega$ is the same as the asymptotic behavior of $v^+ \circ \partial \circ v$. We obtain that the difference between $\omega^+ \circ \partial \circ \omega$ and the standard connection on \mathcal{E}' tends to zero as $\|z\|^{-2}$. (More precisely the entries of corresponding matrices belong to Γ_1^{-2} .)

Section 4.

In this section we'll describe some general properties of instantons on non-commutative spaces. We'll use in our consideration a general construction of

deformations of associative algebra equipped with an action of commutative Lie group; this construction was studied in [11]. We will list some properties of the star-product that can be obtained by the methods of the theory of pseudodifferential operators. These mathematical results permit us to justify some statement of preceding sections.

Let us consider an associative algebra A and a d -dimensional abelian group $L = \mathbf{R}^d$ acting on A by means of automorphisms. This means that for every $v \in \mathbf{R}^d$ we have an automorphism $\alpha_v : A \rightarrow A$ and $\alpha_{v_1+v_2} = \alpha_{v_1} \cdot \alpha_{v_2}$. We assume that A has a norm $\| \cdot \|$ and an involution $*$ that are invariant with respect to automorphisms α_v . It is assumed also that the automorphisms α_v are strongly continuous with respect to v .

One says that an element $a \in A$ is smooth if the function $v \rightarrow \alpha_v(a)$ is infinitely differentiable A -valued function on \mathbf{R}^d (differentiation is defined with respect to the norm $\| \cdot \|$ on A). The set A^∞ of all smooth elements constitutes a subalgebra of A ; the Lie algebra \mathcal{L} of the group L acts on A^∞ by means of infinitesimal automorphisms (derivations).

If θ is a $d \times d$ antisymmetric matrix we can introduce a new product in A^∞ (star-product) by means of the following formula:

$$f \star g = \int \alpha_{\theta u} f \cdot \alpha_v g \cdot e^{iuv} du dv. \quad (26)$$

This new operation is also an associative product and the action of the group $L = \mathbf{R}^d$ preserves it [11]. It depends continuously on θ ; therefore the new algebra A_θ^∞ (the algebra of smooth elements with respect to the product (26)) can be regarded as a deformation of the algebra A^∞ .

Let us consider some examples. One can take as A the algebra $C(T^d)$ of continuous functions on a torus $T^d = \mathbf{R}^d/\mathbf{Z}^d$ equipped with the supremum norm. The Lie group $L = \mathbf{R}^d$ acts on T^d in natural way (by means of translations) and this action generates an action of $L = \mathbf{R}^d$ on $A = C(T^d)$. The algebra A^∞ consists of smooth functions on T^d . The star-product coincides with Moyal product as in all of our examples and A_θ^∞ is interpreted as an algebra of smooth functions on noncommutative torus; we will denote it by T_θ .

Let us take as A the algebra $C_0(\mathbf{R}^d)$ of continuous functions on \mathbf{R}^d that tend to zero at infinity (the norm is again the supremum norm). Using the natural action of $L = \mathbf{R}^d$ on A we obtain an algebra \mathbf{R}_θ^d that can be interpreted as an algebra of smooth functions on noncommutative Euclidean space tending to zero at infinity. We can consider instead of $C_0(\mathbf{R}^d)$ the algebra $C(\mathbf{R}^d \cup \infty)$ of continuous functions on \mathbf{R}^d , that tend to a constant at infinity. Every element of this algebra can be represented in the form $f + \alpha \cdot 1$ where $f \in C_0(\mathbf{R}^d)$. For any algebra A we denote by \tilde{A} the algebra consisting of elements of the form $a + \alpha \cdot 1$ where $a \in A$, $\alpha \in \mathbf{C}$ and 1 is a unit element: $a \cdot 1 = 1 \cdot a = a$. (One says that \tilde{A} is a unitized algebra A .) We see that $C(\mathbf{R}^d \cup \infty) = \widetilde{C(\mathbf{R}^d)}$. The Lie group $L = \mathbf{R}^d$ acts on $C(\mathbf{R}^d)$. Applying the above construction to this

action we obtain a unital algebra $\tilde{\mathbf{R}}_\theta^d$; one can get it adding unit element to the algebra \mathbf{R}_θ^d of the preceding example. Starting with the algebra of all bounded uniformly continuous functions on \mathbf{R}^d with action of translation group $L = \mathbf{R}^d$ we obtain an algebra of bounded functions that have bounded derivatives of all orders (of smooth bounded functions on noncommutative \mathbf{R}^d). It can be identified with the multiplier algebra of \mathbf{R}_θ^d . (By definition the multiplier algebra of A is an endomorphism algebra of A considered as a right A -module.)

It is important to notice that the above construction can be applied also in the case when the topology in the algebra A is specified by a countable family of norms. In particular, we can take $A = S(\mathbf{R}^d)$ (algebra of smooth functions on \mathbf{R}^d tending to zero faster than any power); then the application of the above construction gives an algebra denoted by $S(\mathbf{R}_\theta^d)$. (Notice, that the norms on $S(\mathbf{R}^d)$ are not invariant with respect to translations, therefore formally the results of [11] cannot be applied. Nevertheless, these results remain correct in this case.)

In the case when θ is a nondegenerate matrix d is necessarily even ($d = 2n$). and we identified $S(\mathbf{R}_\theta^d)$ with the algebra of integral operators on \mathbf{R}^n having kernel belonging to $S(\mathbf{R}^d)$.

Various algebras corresponding to noncommutative \mathbf{R}^n are useful in the analysis of various classes of gauge fields. In particular, gauge fields related to \mathbf{R}_θ^d are bounded, gauge fields corresponding to $\tilde{\mathbf{R}}_\theta^d$ tend to zero at infinity, considering $S(\mathbf{R}_\theta^d)$ we allow gauge fields that grow polynomially and in the case of $S(\mathbf{R}_\theta^d)$ we obtain gauge fields that decrease faster than any power at infinity.

Other interesting versions of noncommutative \mathbf{R}^d can be defined by means of methods of the theory of pseudodifferential operators [15]. Recall that using the irreducible representation of commutation relations $[\hat{x}^k, \hat{x}^l] = i\theta^{kl}$, when θ is a nondegenerate $d \times d = 2n \times 2n$ matrix we assign to a function $\varphi(x) = \int \varphi(k) e^{ikx} dk$ an operator $\hat{\varphi}(x) = \int \varphi(k) e^{ik\hat{x}} dk$ acting on functions of n variables. Considering various classes of functions $\varphi(x)$ we obtain various classes of pseudodifferential operators. For example, we can restrict ourselves to the class $\Gamma_\rho^m(\mathbf{R}^d)$ of smooth functions $a(z)$ on \mathbf{R}^d obeying

$$|\partial_\alpha a(z)| \leq C_\alpha \langle z \rangle^{m-\rho|\alpha|}$$

where $\alpha = (\alpha_1, \dots, \alpha_d)$, $m \in \mathbf{R}$, $0 < \rho \leq 1$, $\langle z \rangle = (1 + \|z\|^2)^{1/2}$. One can prove that the star-product of functions $a' \in \Gamma_\rho^{m_1}$ and $a'' \in \Gamma_\rho^{m_2}$ belongs to $\Gamma_\rho^{m_1+m_2}$. If $d = 2n$, θ is nondegenerate then operators corresponding to functions belonging to this class (operators having Weyl symbols from Γ_ρ^m) are called pseudodifferential operators of the class G_ρ^m . It follows from the above statement that the product $A' \cdot A''$ of operators $A' \in G_\rho^{m_1}$ and $A'' \in G_\rho^{m_2}$ belongs to $G_\rho^{m_1+m_2}$. Using this fact it is easy to construct various algebras of pseudodifferential operators.

In particular, Γ_ρ defined as a union of all classes Γ_ρ^m is an algebra with respect to star-product; it corresponds to the algebra of pseudodifferential operators

$G_\rho = \cup_m G_\rho^m$. The class Γ_ρ^m is an algebra for $m \leq 0$. The intersection $\Gamma_\rho^{-\infty}$ of all classes Γ_ρ^m coincides with $S(\mathbf{R}^d)$.

It is important to notice that although algebras \mathbf{R}_θ^d , $S(\mathbf{R}_\theta^d)$, Γ_ρ^m , $m < 0$, are different and are related to different classes of gauge fields they have the same K -theory. For non-degenerate θ these algebras are closely related to the algebra K of compact operators on Hilbert space. One can show that not only elements of K -theory groups, but also projective modules over unitized algebras $\tilde{\mathbf{R}}_\theta^d$, $\tilde{S}(\mathbf{R}_\theta^d)$, $\tilde{\Gamma}_\rho^m$, $m < 0$, are in one-to-one correspondence with projective modules over \tilde{K} . This is the reason why it is possible to use K and \tilde{K} in the consideration of noncommutative solitons and instantons. If K is defined as an algebra of compact operators acting on Hilbert space \mathcal{H} , then the module K^1 can be considered as (completed) direct sum of countable numbers of copies of \mathcal{H} regarded as a K -module. It follows from this representation that $\mathcal{H}^k \oplus K^1$ is isomorphic to K^1 (if we add k elements to a countable set we again obtain a countable set) and that K^n is isomorphic to K^1 . Every projective module over \tilde{K} is isomorphic to a direct sum of several copies of \mathcal{H} and several copies of free module \tilde{K}^1 [see Appendix]. We used analogs of these statements in Sec.2 and 3.

Let us say that a function $a(z)$ belongs to the class $\tilde{\Gamma}_\rho^m(\mathbf{R}^d)$ if $|a(z)| \leq \text{const} \cdot \langle z \rangle^m$, $|\partial_\alpha a(z)| \leq C_\alpha \langle z \rangle^{-\rho|\alpha|}$. It is easy to check that $\tilde{\Gamma}_\rho^m \subset \Gamma_\rho^m$. We'll say that $a \in H\Gamma_\rho^{m,m_0}$ if $a \in \tilde{\Gamma}_\rho^m$ and there exists such a function $b \in \tilde{\Gamma}_\rho^{-m_0}$ that $a \cdot b = 1 - r$, $r \in S(\mathbf{R}^d)$. (In other words, the inverse to the function $a(z)$ with respect to usual multiplication should exist for large $|z|$ and belong to $\tilde{\Gamma}_\rho^{-m_0}$.) One can prove that for a function $a \in H\Gamma_\rho^{m,m_0}$ there exists a function $b_\theta \in H\Gamma_\rho^{-m,-m_0}$ (parametrix) obeying $1 - a \star b_\theta \in S(\mathbf{R}^d)$, $1 - b_\theta \star a \in S(\mathbf{R}^d)$. Here the star-product depends on the noncommutativity parameter θ . (One can say that a function $a(z)$ that is invertible up to an element of $S(\mathbf{R}^d)$ with respect to the usual multiplication has this property also for the star-product.)

Notice that the parametrix is essentially unique: if $b'_\theta \in \Gamma_\rho$ and $1 - a \star b'_\theta \in S(\mathbf{R}^d)$ then $b_\theta - b'_\theta \in S(\mathbf{R}^d)$ (and therefore $1 - b'_\theta \star a \in S(\mathbf{R}^d)$).

Let us assume that pseudodifferential operator \hat{a} corresponding to a function $a \in H\Gamma_\rho^{m,m_0}$ has no zero modes ($\text{Ker } \hat{a} = 0$). Then we can construct an operator $\hat{T} = \hat{a}(\hat{a}^+ \hat{a})^{-1/2}$ obeying $\hat{T}^+ \hat{T} = 1$. It is easy to check that corresponding functions (symbols) T and T^+ belong to $H\Gamma_\rho^{0,0}$. It follows from uniqueness of parametrix that T^+ is a parametrix of T , i. e. $1 - TT^+$ is an integral operator with a kernel from Schwartz space. This remark gives us a way to construct an endomorphism T entering the definition of a field that is gauge trivial at infinity.

The definition of classes Γ_ρ^m , $\tilde{\Gamma}_\rho^m$, $H\Gamma_\rho^{m,m_0}$ was formulated in such a way that it makes sense also for matrix valued functions. (The absolute value should be understood as a matrix norm.) Such functions can be considered as matrices

with entries from Γ_ρ^m ; multiplication of such matrices is defined in terms of star-product and usual matrix multiplication. All statements listed above (including the existence of parametrix) remain true also in the matrix case.

An $s \times s$ matrix A belonging to $H\Gamma_\rho^{m,m_0}$ can be considered as an operator acting on $(\Gamma_\rho)^s$ where $\Gamma_\rho = \cup_m \Gamma_\rho^m$. One can prove that $\text{Ker} A \subset (S(\mathbf{R}_\theta^d))^s$. If θ is a nondegenerate matrix then $\text{Ker} A$, considered as $\tilde{\mathbf{R}}_\theta^d$ -module, is isomorphic to \mathcal{F}^k . (If $A\varphi = 0$, or, more generally, $A\varphi \in (S(\mathbf{R}^d))^s$ it follows from the existence of parametrix that $\varphi \in (S(\mathbf{R}^d))^s$. To check that $\text{Ker} A$ is isomorphic to \mathcal{F}^k it is convenient to consider A as a pseudodifferential operator \hat{A} ; using the parametrix one can derive that \hat{A} is a Fredholm operator. The relation $A\varphi = 0$ implies that $\hat{A}\hat{\varphi} = 0$; in other words $\hat{\varphi}$ is an operator taking values in finite-dimensional vector space $\text{Ker} \hat{A}$. This description leads to identification of $\text{Ker} A$ with \mathcal{F}^k where $k = \dim \text{Ker} \hat{A}$.)

Let us consider an operator \mathcal{A} acting from $(\Gamma_\rho)^{s+t}$ into $(\Gamma_\rho)^s$ and transforming a pair (u, v) where $u \in (\Gamma_\rho)^s$, $v \in (\Gamma_\rho)^t$ into $Au + Cv \in (\Gamma_\rho)^s$. Let us suppose that the operator A is represented by an $s \times s$ matrix, belonging to $H\Gamma_\rho^{m,m_0}$, where $m > 0$, and C is represented by an $s \times t$ matrix belonging to $(\Gamma_\rho)^0$. We denote by Q a parametrix of A (i.e. $QA = 1 - \Pi$, $AQ = 1 - \Pi'$, and $\Pi, \Pi' \in S(\mathbf{R}^d)$).

We would like to construct an \mathbf{R}_θ^d -linear isomorphism between \mathbf{R}_θ^d -modules $\mathcal{E} = \text{Ker} \mathcal{A} \cap (\mathbf{R}_\theta^d)^{s+t}$ and $\mathcal{E}' = \mathcal{F}^k \oplus (\mathbf{R}_\theta^d)^t$. Let us assume first that $QA = 1$ (i.e. $\Pi' = 0$). Then $Au + Cv = 0$ implies $u = u_0 - QCv$ where $u_0 \in \text{Ker} A = \mathcal{F}^k$. Let us check that the map ν transforming (ρ, v) into $(\rho - QCv, v)$ is an isomorphism between \mathcal{E}' and \mathcal{E} . This map is obviously \mathbf{R}_θ^d -linear and has trivial kernel. To check that the image of a point $(\rho, v) \in \mathcal{E}'$ belongs to \mathcal{E} we notice that $Q \in H\Gamma_\rho^{-m, -m_0} \subset \Gamma_\rho^{-m}$ and therefore $QC \in \Gamma_\rho^{-m}$. We assumed that $m > 0$, hence $-m < 0$ and $v \in (\mathbf{R}_\theta^d)^s$ implies $QCv \in (\mathbf{R}_\theta^d)^s$. Conversely, the map $\nu^{-1} : (u, v) \rightarrow (u + QCv, v)$ transforms \mathcal{E} into \mathcal{E}' .

The above consideration clarifies and generalizes the arguments applied in Sec.3 to the case when $A \in H\Gamma_1^{1,1}$ and $C \in \Gamma_1^0$ are specified by the formula (23).

Now we should get rid of the condition $\Pi' = 0$. This can be done by means of change of variables: $\tilde{u} = u$, $\tilde{v} = v + Mu$, where operator M is selected in such a way that the above arguments can be applied to the equation $(A - CM)\tilde{u} + C\tilde{v} = 0$.

If a commutative Lie algebra $\mathcal{L} = \mathbf{R}^d$ acts on associative algebra A by means of derivations we can define a connection on a (right) A -module E as a collection of \mathbf{C} -linear operators $\nabla_1, \dots, \nabla_d$ acting on E and obeying the Leibniz rule:

$$\nabla_i(ea) = \nabla_i e \cdot a + e\delta_i a \quad (27)$$

where $e \in E$, $a \in A$, $\delta_i : A \rightarrow A$ are derivations (infinitesimal automorphisms) corresponding to elements of a basis of \mathcal{L} . A curvature of a connection (field strength of noncommutative gauge field) is defined by the formula

$$F_{ij} = [\nabla_i, \nabla_j] \quad (28)$$

It is easy to check that $F_{ij} : A \rightarrow A$ are linear maps (endomorphisms of A -module E). One can consider the curvature as a 2-form F on $\mathcal{L} = \mathbf{R}^d$ taking values in the algebra of endomorphisms $\text{End}_A E$.

We'll work with involutive algebras and Hilbert modules (i.e. we assume that the algebra A is equipped with antilinear involution $a \rightarrow a^*$ and the modules are provided with A -valued inner product). Then it is natural to consider gauge fields as unitary connections; the endomorphisms F_{ij} will be antihermitian operators in this case.

If this algebra is equipped with a trace and $\mathcal{L} = \mathbf{R}^d$ is equipped with a non-degenerate inner product (with metric tensor g_{ij}) then we can define Yang-Mills action functional on connection on the A -module E

$$S = \text{Tr} F_{ij} F^{ij} = \text{Tr} \langle F, F \rangle. \quad (29)$$

(The indices are raised and lowered by means of metric tensor g_{ij} .)

In particular, we can apply the construction of action functional to the cases when $A = T_\theta^d, \mathbf{R}_\theta^d$ or $\tilde{\mathbf{R}}_\theta^d$.

Using the metric tensor g_{ij} on \mathcal{L} we can define the Hodge dual of an exterior form on \mathcal{L} . In the case $d = 4$ we can define an instanton (antiinstanton) as a gauge field obeying

$$F \pm *F = \omega \cdot 1 \quad (30)$$

where ω is a \mathbf{C} -valued 2-form on \mathcal{L} . In other words, the self-dual part of the curvature of an instanton should be a scalar (or, more precisely, a scalar multiple of unit endomorphism). In the case of an antiinstanton this condition should be fulfilled for the antiselfdual part of the curvature. We suppose that the metric on \mathcal{L} used in the definition of (anti)instanton is positive.

Let us introduce a complex structure on $\mathcal{L} = \mathbf{R}^4$ in such a way that \mathcal{L} becomes a Kaehler manifold. Without loss of generality we can assume the metric has the form $ds^2 = dz_1 d\bar{z}_1 + dz_2 d\bar{z}_2$ in complex coordinates z_1, z_2 on \mathcal{L} . Then instead of operators $\nabla_1, \nabla_2, \nabla_3, \nabla_4$ corresponding to the coordinates x_1, x_2, x_3, x_4 on \mathcal{L} it is convenient to consider operators $D_1 = \nabla_1 + i\nabla_2, D_2 = \nabla_3 + i\nabla_4, D_1^* = -\nabla_1 + i\nabla_2, D_2^* = -\nabla_3 + i\nabla_4$. (Here $z_1 = x_1 + ix_2, z_2 = x_3 + ix_4$.) The instanton equation $F + *F = \omega \cdot 1$ can be represented in components

$$F_{11} + F_{34} = \omega_{11} \cdot 1$$

$$F_{14} + F_{23} = \omega_{14} \cdot 1$$

$$F_{13} + F_{42} = \omega_{13} \cdot 1$$

or, equivalently, in the form

$$[D_1, D_2] = \lambda \cdot 1 \quad (31)$$

$$[D_1, D_1^+] + [D_2, D_2^+] = \mu \cdot 1. \quad (32)$$

In noncommutative case one can describe the moduli space of instantons in terms of holomorphic vector bundles (Donaldson's theorem). There exists an analog of this description for noncommutative instantons; the role of holomorphic bundles is played by solutions of (31).

Let us consider the case when $A = T_\theta^d$ and E is a projective module (a direct summand in a free module). There exists a natural trace on T_θ^d and this trace induces a trace on the algebra of endomorphisms $\text{End}_A E$. (Endomorphisms of a free module over a unital algebra A can be represented as matrices with entries from A . Trace of such a matrix can be defined as a sum of traces of diagonal elements. An endomorphism of a projective module can be extended to an endomorphism of a free module. This permits us to define a trace of an endomorphism of projective module.)

It is easy to check that $\text{Tr} F_{ab}$ does not depend on the choice of connection ∇ on T_θ^d -module E . To derive this statement we can use the formula for infinitesimal variation of F_{ab} :

$$\delta F_{ab} = [\nabla_a, \delta \nabla_b] + [\delta \nabla_a, \nabla_b].$$

It remains to use the fact that

$$\text{Tr}[\nabla_a, \varphi] = 0 \quad (33)$$

for every endomorphism φ . (By definition, the trace vanishes on commutator of endomorphisms. The difference $\nabla'_a - \nabla_a$ where ∇' , ∇ are two connections is an endomorphism, hence it is sufficient to check (32) only for one connection. For example, one can check (32) for so called Levi-Civita connection $P\delta_i P$ where P is a projection of free module on the projective module E .) We see that $\text{Tr} F_{ab}$ is a topological number characterizing the T_θ^d -module E . Other numbers of this kind can be obtained by the formula $\text{Tr} F^n$ where F is considered as a 2-form on \mathcal{L} . To prove that $\text{Tr} F^n$ does not depend on the choice of connection we again should use (33). If $d = 4$ we have the following topological numbers: $\text{Tr} 1$ (noncommutative dimension), $\text{Tr} F_{ab}$ (magnetic flux) and $\text{Tr} F^2$. For the case of commutative torus $(2\pi)^{-n} \text{Tr} F^n$ is an integer. In the case of noncommutative torus T_θ^d also one can find integer numbers characterizing topological class (more precisely, K -theory class) of the module E : as in commutative case we have integer-valued antisymmetric tensors of even rank: $\mu^{(0)}, \mu_{ij}^{(2)}, \mu_{ijkl}^{(4)}, \dots$ (K -theory group is a discrete object; it does not change by continuous variation of θ). These numbers can be related to $\text{Tr} F^n$.

One can prove that the value of action functional (29) on a gauge field obeying (30) (on an (anti)instanton) can be expressed in terms of topological numbers of the module E and that this value gives a minimum to the action functional on gauge fields on E . The proof is based on the inequality

$$\text{Tr}(F \pm *F - \omega \cdot 1)(F \pm *F - \omega \cdot 1)^* \geq 0 \quad (34)$$

that follows from the positivity of trace (from relation $\text{Tr} A^* A \geq 0$). The left hand side of (34) can be expressed in terms of the value of action functional and of topological numbers of the module E ; this gives the estimate we need (see [7] for detail). The above consideration looks very general; however, we encounter some problems trying to apply it to \mathbf{R}_θ^4 or to $\tilde{\mathbf{R}}_\theta^4$. First of all, one can define a trace of element $f \in \mathbf{R}_\theta^4$ as an integral of f over \mathbf{R}^4 , but this integral can be divergent. This trace is positive; we can extend it to a trace on $\tilde{\mathbf{R}}_\theta^4$ assigning an arbitrary value to $\text{Tr} 1$, but the extended trace is not positive. Therefore in Sec 1 and 2 we defined instantons with $\omega = 0$ in the right hand side of (30). However, the topological number $\text{Tr} F^2$ is well defined for instantons on \mathbf{R}_θ^4 . (This follows from finiteness of euclidean action.) The above arguments show that the euclidean action of an instanton on \mathbf{R}_θ^4 is minimal in the set of gauge fields with given topological number $\text{Tr} F^2$. It is important to notice that the topological number $\text{Tr} F$ is ill-defined in this situation (the corresponding integral diverges).

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Appendix

Projective modules over \tilde{K}

Let us consider the C^* -algebra K of compact operators on Hilbert space \mathcal{H} and corresponding unitized algebra \tilde{K} . One can prove the following statement.

Every finitely generated projective module over \tilde{K} is isomorphic to a direct sum of several copies of \mathcal{H} and several copies of free module \tilde{K}^1 .

Proof. Let ρ be the canonical character of \tilde{K} (i.e. the canonical map $\tilde{K} \rightarrow \tilde{K}/K = \mathbf{C}$). Let $e \in M_q(\tilde{K})$ be a selfadjoint idempotent and $\rho(e) \in M_q(\mathbf{C})$ the corresponding scalar idempotent. The density of finite rank operators in K shows that for any $\epsilon > 0$ one can find a closed subspace E of finite codimension in \mathcal{H} and a selfadjoint $x \in M_q(\tilde{K})$ such that the norm of $x - e$ is less than ϵ and the matrix elements of $x - \rho(e)$ all vanish on E . Viewing x as an element of $M_q(\tilde{A})$ where A is the matrix algebra of operators in the orthogonal of E one gets that for $\epsilon > 0$ small enough, x is close to a self-adjoint idempotent $f \in M_q(\tilde{A})$ and that f is equivalent to e in $M_q(\tilde{K})$. The result then follows from the trivial determination of the finite projective modules on \tilde{A} .

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